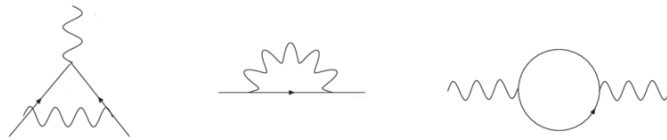


One-loop corrections in quantum electrodynamics

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Abstract

In this project, students get – on a practical level – familiar with the concept of renormalization and the computation of quantum corrections in a relativistic quantum field theory, which allows to make predictions for physical observables beyond the Born approximation.

The goal of this project is to derive the one-loop renormalization constants of quantum electrodynamics (QED), and to use them for the computation of the one-loop correction to the anomalous magnetic moment of the electron. For both tasks a computer-based approach is envisaged, based on the programming language *Mathematica*.

As a prerequisite the students should have attended Theoretical Particle Physics 1 and 2. The lecture on loop integrals is beneficial but not mandatory. Also basic knowledge about a computer algebra program such as *Mathematica* is beneficial.

Prior to starting this project

- Contact the supervisor about access to *Mathematica*.

1 Introduction

The computation of perturbative corrections to particle reactions in the framework of quantum field theory plays an essential role in contemporary particle physics research. The main goal is to increase the precision of theoretical predictions for particle reactions and to compare them to experimental measurements, whose statistical uncertainties also decrease as more and more data becomes available. In this way, one hopes to disentangle small deviations between theory and experiment, which might hint for new, yet undiscovered fundamental physical effects.

Historically, it turned out to be extremely difficult to get access to quantum corrections in relativistic field theory since it was unclear how to consistently formulate the relativistic quantum field theory of electrons and photons, and how to handle the occurring divergencies when computing quantum corrections. The breakthrough was achieved by Feynman, Schwinger and Tomonaga in the 1940s [1–4], and the $\mathcal{O}(\alpha)$ -correction to the anomalous magnetic moment g_e of the electron represented a seminal milestone [5]. Ever since, corrections of higher order in the fine structure constant $\alpha \sim 1/137$ have become available, and resulted in a striking agreement between theory and experiment, as the following numbers for $a_e = (g_e - 2)/2$ show.

$$\begin{aligned} a_e &= 0 && \text{(leading order)} \\ a_e &= \frac{\alpha}{2\pi} \approx 0.0011614 && \text{(to } \mathcal{O}(\alpha) \text{) [5]} \\ a_e &= 0.001159652181643(764) && \text{(to } \mathcal{O}(\alpha^5) \text{) [6, 7]} \\ a_e &= 0.00115965218073(28) && \text{(experimental value) [8] .} \end{aligned}$$

Since then, countless quantum corrections to particle-physics observables have become available, and have confirmed the need of these contributions to describe experimental data accurately. In some cases, however, discrepancies between theory and experiment persist, as is currently the case for the anomalous magnetic moment $a_\mu = (g_\mu - 2)/2$ of the muon [9–11], see left panel of figure 1. Here, there are hadronic loop corrections which are hard to calculate (last two diagrams of figure 1) and which are subject to contemporary research.

Only for the simplest cases it is feasible to perform a loop calculation by hand. Therefore, the computer became an indispensable tool in carrying out loop calculations in quantum field theory. Many dedicated programs were developed, tailored for particular steps of the calculation or for carrying out the calculation as a whole.

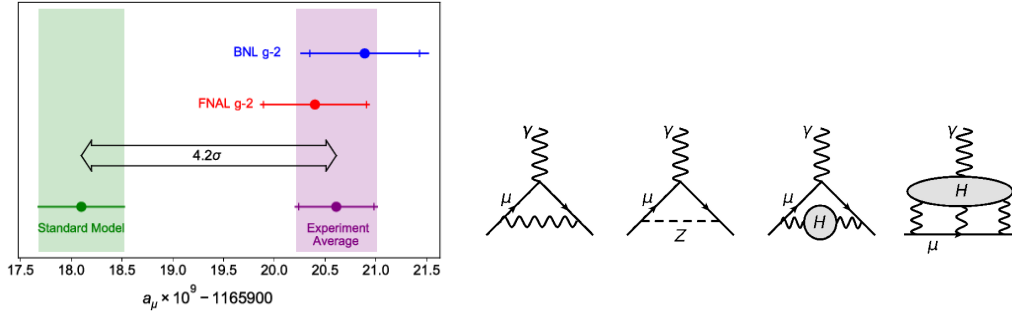


Figure 1: The anomalous magnetic moment of the muon [9–11].

In this project, we first get familiar with the theory of quantum electrodynamics (QED), the basics of regularization (in particular dimensional regularization) and renormalization, and the Mathematica package `FeynCalc`, which allows to automate many one-loop calculations. The task will then be to use the `FeynCalc` package to renormalize QED at the one-loop level in the so-called pole scheme, and to obtain the one-loop correction to the anomalous magnetic moment of the electron.

2 Quantum electrodynamics

Quantum electrodynamics (QED) is a relativistic, quantum-field-theoretic description of the electromagnetic interaction of electrons and positrons with photons.

2.1 Lagrangian density, gauge symmetry

The Lagrangian density of QED reads

$$\mathcal{L} = \underbrace{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}}_{\text{free photon field}} + \underbrace{\bar{\psi}(i\not{\partial} - m)\psi}_{\text{free electron field}} + \underbrace{e\bar{\psi}\not{A}\psi}_{\text{interaction}}.$$

Here ψ is a Dirac spinor describing the fermion (electron, positron), A_μ is a vector field describing the photon, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength tensor. m is the fermion mass, and $e = \sqrt{4\pi\alpha}$ the electric charge of the positron ($\alpha \sim 1/137$ is the fine structure constant). In \mathcal{L} we distinguish terms that describe free fields from interaction terms. The former contain

at most two fields, and are classified as mass terms (with masses) or kinetic terms (with derivatives). The interaction terms have more than two fields.

The Lagrangian density of QED is invariant under the local gauge transformation

$$\begin{aligned}\psi(x) &\rightarrow \psi'(x) = e^{ie\Lambda(x)} \psi(x), \\ A_\mu &\rightarrow A'_\mu(x) = A_\mu + \partial_\mu \Lambda(x).\end{aligned}$$

Moreover, the interaction term can be obtained from the theory of free fields by means of a procedure called “minimal substitution”,

$$\partial_\mu \rightarrow D_\mu = \partial_\mu - ieA_\mu(x).$$

D_μ is called the “covariant derivative”.

2.2 Feynman rules

The Feynman rules that can be derived from the Lagrangian density are classified as propagators, external lines and vertices, and are usually given in momentum space.

The propagators are the Green’s functions of the free field equations ($e = 0$), e.g. for the electron

$$(i\not{\partial} - m)G(x) = \delta^{(4)}(x),$$

which after Fourier transformation

$$G(x) = \int \frac{d^4p}{(2\pi)^4} e^{-ipx} \tilde{G}(p), \quad \delta^{(4)}(x) = \int \frac{d^4p}{(2\pi)^4} e^{-ipx}$$

gives

$$\tilde{G}(p) = \frac{\not{p} + m}{p^2 - m^2 + i\eta}.$$

The electron propagator is then $\tilde{S}(p) = i\tilde{G}(p)$, and the infinitesimal $\eta > 0$ moves the poles away from the real axis such as to have causal boundary conditions. The attempt to perform analogous steps to derive the photon

propagator $D^{\nu\rho}(x) = \int \frac{d^4p}{(2\pi)^4} e^{-ipx} \tilde{D}^{\nu\rho}(p)$ from

$$(-g_{\mu\nu}\square + \partial_\mu\partial_\nu)D^{\nu\rho}(x) = -\delta_\mu^\rho \delta^{(4)}(x)$$

fails since the equation

$$(p^2 g_{\mu\nu} - p_\mu p_\nu) \tilde{D}^{\nu\rho}(p) = -\delta_\mu^\rho$$

cannot be inverted ($M_{\mu\nu} = p^2 g_{\mu\nu} - p_\mu p_\nu$ has an eigenvalue zero due to $p^\mu M_{\mu\nu} = 0$). One possible solution is to add a gauge-fixing term to \mathcal{L} ,

$$\mathcal{L} \rightarrow \mathcal{L} - \frac{1}{2\xi} (\partial_\mu A^\mu)^2,$$

which then gives

$$\tilde{D}^{\nu\rho}(p) = -\frac{g^{\nu\rho} - (1 - \xi) \frac{p^\nu p^\rho}{p^2 + i\eta}}{p^2 + i\eta}.$$

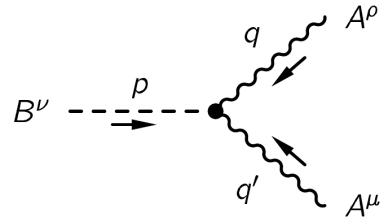
The photon propagator is then $\tilde{\Delta}^{\nu\rho}(p) = i \tilde{D}^{\nu\rho}(p)$. ξ is called *gauge parameter*. Physical observables do not depend on ξ . One of the common choices is the *Feynman gauge* $\xi = 1$ which we will use throughout.

The Feynman rules for the external lines can simply be taken from the classical solution of the free field equations. For the electron/positron we get the familiar spinors $u(p, s)$, $\bar{u}(p, s)$, $v(p, s)$, $\bar{v}(p, s)$ for incoming/outgoing (anti)fermion. For the photon we obtain the polarization vector $\epsilon^\mu(p, \lambda)$.

In order to derive Feynman rules for interaction vertices from \mathcal{L} we first give the general prescription, together with an example, and subsequently apply it to QED. The recipe consists of four steps

1. Search for all products in \mathcal{L} containing a previously specified combination of fields. These form the external lines of the vertex.

$$\text{E.g. } -g(\partial_\mu A_\nu) A^\mu B^\nu = -g(\partial_\mu A^\rho) g_{\rho\nu} A^\nu B^\mu$$



2. Replace all derivatives by $(-i)$ times the incoming momenta of the fields on which they act.

$$\longrightarrow -g(-iq_\mu) g_{\rho\nu} A^\rho A^\nu B^\mu = ig(q_\mu g_{\rho\nu}) A^\rho A^\nu B^\mu$$

3. Sum over all permutations of the indices and momenta of identical external fields. (Note that ψ and $\bar{\psi}$ are not identical fields!)

$$\longrightarrow ig(q_\mu g_{\rho\nu} + q'_\rho g_{\mu\nu}) A^\rho A^\nu B^\mu$$

4. Discard all external fields.

$$\longrightarrow ig(q_\mu g_{\rho\nu} + q'_\rho g_{\mu\nu}) \quad .$$

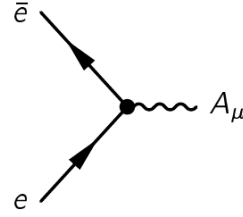
Applied to QED we find

1. $ie\bar{\psi}\gamma_\mu\psi A^\mu$

2. $ie\bar{\psi}\gamma_\mu\psi A^\mu$

3. $ie\bar{\psi}\gamma_\mu\psi A^\mu$

4. $ie\gamma_\mu$.



These rules are supplemented by the following items

- Impose four-momentum conservation at each vertex
- Integrate over each unconstrained (loop) momentum with measure $\int \frac{d^4k}{(2\pi)^4}$
- Along a fermion line, order the Dirac matrices and spinors opposite to the direction of fermion flow.
- Include a factor (-1) and a Dirac trace for each closed fermion loop.
- Include a factor (-1) when permutating external fermion lines in processes with more than one diagram.

More on Feynman rules in QED and other theories can be found in the literature on quantum field theory, e.g. [12–16]). We summarize the QED Feynman rules below.

2.3 QED Feynman rules, summary

$\frac{p}{\longrightarrow} = \frac{i(\not{p} + m)}{p^2 - m^2 + i\eta}$	$\frac{p,s}{\longrightarrow \bullet} = u(p, s)$	$\frac{p,s}{\bullet \longrightarrow} = \bar{u}(p, s)$
$\mu \frac{p}{\sim\sim\sim} \nu = -i \frac{g^{\mu\nu} - (1 - \xi) \frac{p^\mu p^\nu}{p^2 + i\eta}}{p^2 + i\eta}$	$\frac{p,s}{\longleftarrow \bullet} = \bar{v}(p, s)$	$\frac{p,s}{\bullet \longleftarrow} = v(p, s)$
$\mu \begin{array}{c} \nearrow \\ \sim\sim\sim \\ \searrow \end{array} = ie\gamma^\mu$	$\frac{p,\lambda}{\sim\sim\sim \bullet} = \epsilon_\mu(p, \lambda)$	$\frac{p,\lambda}{\bullet \sim\sim\sim} = \epsilon_\mu^*(p, \lambda)$

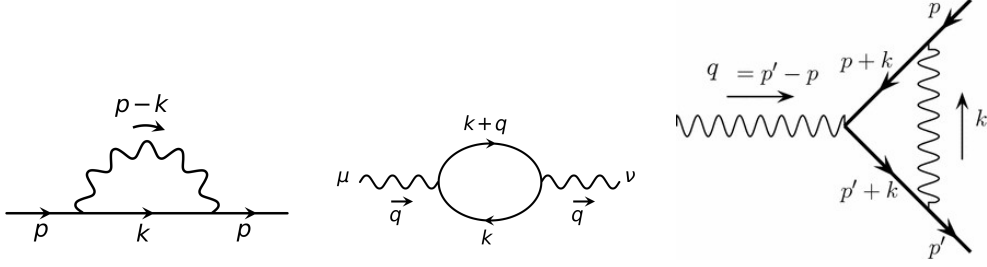


Figure 2: One-particle irreducible one-loop diagrams in QED.

2.4 One-loop QED diagrams

In order to renormalize QED at the one-loop level, we need to consider the one-particle irreducible (1PI) one-loop diagrams of QED, which are the electron self energy (left panel of figure 2), the vacuum polarization (middle), and the vertex correction (right). The 1PI diagrams are defined without the external lines. Let us apply the QED Feynman rules to the three cases in turn.

Electron self-energy The amplitude reads

$$-i\Sigma_{1L} = \int \frac{d^4k}{(2\pi)^4} (ie\gamma_\mu) \frac{i(\not{k} + m)}{[k^2 - m^2 + i\eta]} (ie\gamma_\nu) \frac{-ig^{\mu\nu}}{[(k-p)^2 + i\eta]}$$

Vacuum polarization The amplitude reads

$$i\Pi_{1L}^{\mu\nu} = -Tr \left[\int \frac{d^4k}{(2\pi)^4} (ie\gamma^\mu) \frac{i(\not{k} + m)}{[k^2 - m^2 + i\eta]} (ie\gamma^\nu) \frac{i(\not{k} + \not{q} + m)}{[(k+q)^2 - m^2 + i\eta]} \right]$$

Vertex correction The amplitude reads

$$\begin{aligned} \Lambda_{1L}^\mu &= \int \frac{d^4k}{(2\pi)^4} (ie\gamma_\rho) \frac{i(\not{k} + \not{p}' + m)}{[(k+p')^2 - m^2 + i\eta]} (ie\gamma^\mu) \frac{i(\not{k} + \not{p} + m)}{[(k+p)^2 - m^2 + i\eta]} \\ &\quad \times (ie\gamma_\sigma) \frac{-ig^{\rho\sigma}}{[k^2 + i\eta]} \end{aligned}$$

We will see below that we have to slightly modify these expressions in the framework of dimensional regularization.

3 Regularization and renormalization

The integration over the loop momentum k^μ with integration measure $\int \frac{d^4 k}{(2\pi)^4}$ oftentimes results in divergences as the components $|k^\mu| \rightarrow \infty$, so-called UV-divergences. In order to make higher orders in perturbation theory manageable, one therefore has to

1. Extract the divergent and the finite pieces of each integral, preferably in a way that does not spoil the underlying symmetries (Lorentz invariance, gauge invariance, ...). This procedure is called *regularization*.
2. Absorb the extracted infinities into so-called counterterms (see below) to make the results of scattering amplitudes finite also beyond the Born-level. This procedure is called *renormalization*.

We leave the formal aspects of renormalization to the quantum-field theory literature [12–16]) and take a practical approach here.

3.1 Dimensional regularization

In contemporary particle physics the most-applied regularization scheme is *dimensional regularization* (*dim. reg.*). The idea is to perform a transition from $D = 4$ to $D = 4 - 2\epsilon$ (ϵ small) space-time dimensions according to well-specified rules given below. The divergences of a loop integral then occur as poles in ϵ , together with finite parts and higher order in ϵ . The main advantages of dimensional regularization are that it conserves all underlying symmetries, that it can be analytically continued to complex values of D , and that many relations can be algorithmically implemented on a computer. A drawback arises in calculations involving the Dirac matrix γ_5 .

How is the transition from $D = 4$ to $D = 4 - 2\epsilon$ realized in practice?

- In loop integrals, replace $\int \frac{d^4 k}{(2\pi)^4} \longrightarrow \int \frac{d^D k}{(2\pi)^D}$.
- In phase-space integrals, replace

$$\begin{aligned} \int \frac{d^3 \vec{p}}{(2\pi)^3 2p^0} &\longrightarrow \int \frac{d^{D-1} \vec{p}}{(2\pi)^{D-1} 2p^0} , \\ (2\pi)^4 \delta^{(4)}(p_f - p_i) &\longrightarrow (2\pi)^D \delta^{(D)}(p_f - p_i) \end{aligned}$$

- In the calculation with Lorentz indices, one has to take into account that $g^\mu_\mu = D$. Other than that, the calculation with Lorentz indices

remains basically unchanged, e.g. raising and lowering indices work as before.

- There are also consequences on the Dirac algebra. While the form of the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ remains unchanged, the appearance of the D -dimensional metric on the r.h.s. alters the contractions,

$$\begin{aligned}\gamma^\mu \gamma_\mu &= D, \\ \gamma^\mu \gamma^\alpha \gamma_\mu &= (2 - D) \gamma^\alpha, \\ \gamma^\mu \gamma^\alpha \gamma^\beta \gamma_\mu &= 2\gamma^\beta \gamma^\alpha - (2 - D) \gamma^\alpha \gamma^\beta\end{aligned}$$

etc. For the computation of Dirac traces one *defines* $Tr(\mathbb{1}) = 4$, which makes traces such as $Tr(\gamma^\mu \gamma^\nu) = 4g^{\mu\nu}$ look as before, with the D -dimensional metric appearing again on the r.h.s.

- The canonical (mass) dimension of various quantities changes. The action must remain dimensionless since we want to make it minimal/stationary due to the principle of least action, $[S] = 0$. $[x^\mu] = -1$ and $[m] = 1$ also remain unchanged.

Q: Using $S = \int d^D x \mathcal{L}$ with the QED Lagrangian $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\not{\partial} - m)\psi + e \bar{\psi} \not{A} \psi$, what is the canonical dimension of \mathcal{L} , A_μ , ψ , e ? What is the canonical dimension of a scalar field $\phi(x)$?

- Scaleless integrals vanish in dimensional regularization, e.g. $\int \frac{d^D k}{(2\pi)^D} (k^2)^\alpha = 0$.

3.2 The counterterm method

The essence of the counterterm method is to treat the quantities in the Lagrangian density as so-called *bare quantities*, indicated by and index 0,

$$\begin{aligned}\mathcal{L}_{0, \text{QED}} &= -\frac{1}{4} F_0^{\mu\nu} F_{\mu\nu}^0 + \bar{\psi}_0(i\not{\partial} - m_0)\psi_0 + e_0 \bar{\psi}_0 \not{A}_0 \psi_0 \\ &= -\frac{1}{4} (\partial_\mu A_\nu^0 - \partial_\nu A_\mu^0) (\partial^\mu A_0^\nu - \partial^\nu A_0^\mu) + \bar{\psi}_0(i\not{\partial} - m_0)\psi_0 + e_0 \bar{\psi}_0 \not{A}_0 \psi_0.\end{aligned}$$

Here A_μ^0 and ψ_0 are the bare fields, m_0 is the bare mass, and e_0 is the bare coupling. The gauge fixing term is not relevant for this particular discussion. We now relate the bare quantities to so-called *renormalized quantities* (without index 0) via multiplicative *Z-factors*,

$$\begin{aligned}
A_0^\mu &= \sqrt{Z_3} A^\mu, & m_0 &= Z_m m, \\
\psi_0 &= \sqrt{Z_2} \psi, & e_0 &= \frac{Z_1}{Z_2 \sqrt{Z_3}} e \tilde{\mu}^\epsilon.
\end{aligned}$$

The reason for introducing Z_1 in the specific combination of the last equation will become clear later on. Into the Z -factors we will absorb the infinities arising from loop calculations according to certain well-defined rules (= *renormalization schemes*), to be specified below. In the above equations we also introduced the quantity $\tilde{\mu}$ which is related to the so-called *renormalization scale* μ via

$$\tilde{\mu}^2 = \mu^2 e^{\gamma_E - \ln(4\pi)}.$$

$\gamma_E = \lim_{n \rightarrow \infty} [-\ln(n) + \sum_{k=1}^n \frac{1}{k}] \simeq 0.5772$ is the Euler-Mascheroni constant. Both μ and $\tilde{\mu}$ have mass-dimension +1 and are required to make the renormalized coupling dimensionless. Further useful relations in this context are

$$\tilde{\mu}^{2\epsilon} = \frac{1}{(4\pi)^\epsilon} e^{\epsilon(\gamma_E + \ln(\mu^2))}, \quad \frac{\tilde{\mu}^{2\epsilon}}{(4\pi)^{D/2}} = \frac{1}{(4\pi)^2} e^{\epsilon(\gamma_E + \ln(\mu^2))}.$$

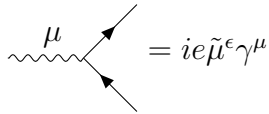
We now write the Lagrangian in terms of the renormalized fields,

$$\mathcal{L}_{0, \text{QED}} = -\frac{1}{4} Z_3 F^{\mu\nu} F_{\mu\nu} + Z_2 \bar{\psi} i \not{\partial} \psi - Z_2 Z_m m \bar{\psi} \psi + e \tilde{\mu}^\epsilon Z_1 \bar{\psi} \not{A} \psi$$

and split it up into

$$\begin{aligned}
\mathcal{L}_{0, \text{QED}} &= -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \bar{\psi} (i \not{\partial} - m) \psi + e \tilde{\mu}^\epsilon \bar{\psi} \not{A} \psi \\
&- \frac{1}{4} (Z_3 - 1) F^{\mu\nu} F_{\mu\nu} + (Z_2 - 1) \bar{\psi} i \not{\partial} \psi - (Z_2 Z_m - 1) m \bar{\psi} \psi + (Z_1 - 1) e \tilde{\mu}^\epsilon \bar{\psi} \not{A} \psi.
\end{aligned}$$

The first term is the usual QED Lagrangian, expressed in terms of renormalized quantities. It is used to derive propagators and vertices in the usual way. While the rules for propagators and external lines remain unchanged, the vertex now becomes



$$= ie \tilde{\mu}^\epsilon \gamma^\mu$$

The second line in the above equation is the so-called counterterm Lagrangian \mathcal{L}_{CT} . It is treated as an interaction Lagrangian although it contains terms that have fewer than three fields. Its Feynman rules are

$$\begin{array}{c} \xrightarrow{p} \quad \xrightarrow{p} \\ \text{---} \times \text{---} \end{array} = i [(Z_2 - 1)\not{p} - (Z_2 Z_m - 1)m]$$

$$\begin{array}{c} \mu \quad p \quad p \quad \nu \\ \text{---} \times \text{---} \end{array} = -i(Z_3 - 1) [p^2 g^{\mu\nu} - p^\mu p^\nu]$$

$$\begin{array}{c} \mu \\ \text{---} \times \end{array} = i(Z_1 - 1)e\gamma^\mu$$

The Z -factors are expanded in powers of α according to

$$Z_i = 1 + \frac{\alpha}{4\pi} \delta Z_i^{(1)} + \left(\frac{\alpha}{4\pi}\right)^2 \delta Z_i^{(2)} + \dots$$

3.3 Renormalization schemes

There exist several renormalization prescriptions, referred to as *renormalization schemes*, of which we summarize two of the most common ones below.

Pole scheme (on-shell scheme). The condition for the on-shell scheme is that the 1PI two-point function has, in the vicinity of the physically meaningful (on-shell) point, the same form as in the free theory.

In practice this means that for the electron self-energy we must require that the overall one-loop correction around the point $\not{p} = m$ vanishes up to linear order in $(\not{p} - m)$, which gives

$$\Sigma_{1L} + \Sigma_{CT} = \underbrace{(\Sigma_{1L} + \Sigma_{CT})|_{\not{p}=m}}_{\stackrel{!}{=} 0} + \underbrace{\frac{\partial}{\partial \not{p}}(\Sigma_{1L} + \Sigma_{CT})|_{\not{p}=m}}_{\stackrel{!}{=} 0} (\not{p} - m) + \mathcal{O}((\not{p} - m)^2)$$

Σ_{CT} is the counterterm contribution at $\mathcal{O}(e^2)$.

In the case of the vacuum polarization the on-shell point is at $q^2 = 0$. Writing $\Pi^{\mu\nu} = (g^{\mu\nu} q^2 - q^\mu q^\nu) \Pi(q^2)$ (this form is required by gauge invariance), the on-shell renormalization condition reads

$$\Pi_{1L} + \Pi_{CT}|_{q^2=0} \stackrel{!}{=} 0.$$

Finally the vertex correction at the on-shell point $p^2 = p'^2 = m^2$, $q = 0$ can be related to the counterterm of the electron self-energy, resulting in

$$\Lambda_{1L}^{\mu, OS} = -ie\tilde{\mu}^\epsilon (Z_2 - 1) \gamma^\mu.$$

Together with $\Lambda_{CT}^\mu = ie\tilde{\mu}^\epsilon (Z_1 - 1) \gamma^\mu$ and the condition $\Lambda_{1L}^\mu + \Lambda_{CT}^\mu = 0$, we derive the relation $Z_1 = Z_2$. This identity is called *Ward identity* and is a consequence of gauge invariance. We derived it to one-loop order and in the on-shell scheme, but it holds to any order and in any renormalization scheme (as long as Z_1 and Z_2 are renormalized in the same scheme).

$\overline{\text{MS}}$ scheme. In the *modified minimal subtraction scheme* we subtract only divergences, no finite or higher parts in the ϵ -expansion. The Z -factors therefore contain only pole-terms in ϵ . The three conditions read

$$\Sigma_{1L} + \Sigma_{CT} \stackrel{!}{=} \text{finite},$$

$$\Pi_{1L} + \Pi_{CT} \stackrel{!}{=} \text{finite},$$

$$\Lambda_{1L}^\mu + \Lambda_{CT}^\mu \stackrel{!}{=} \text{finite}.$$

4 Loop integrals

We summarize here some of the techniques to calculate loop integrals. The simplest non-trivial one-loop integral is

$$I_n(m^2) = \int \frac{d^D k}{(2\pi)^D} \frac{1}{[k^2 - m^2 + i\eta]^n}$$

with an infinitesimal quantity $\eta > 0$ to shift the poles of the propagator away from the real axis. We look at the k_0 -integration first,

$$I_n(m^2) = \int \frac{d^{D-1} k}{(2\pi)^D} \int dk_0 \frac{1}{[k_0^2 - \vec{k}^2 - m^2 + i\eta]^n}$$

and note that the poles are at $k_0^\pm = \pm \sqrt{\vec{k}^2 + m^2 - i\eta} = \pm \sqrt{\vec{k}^2 + m^2} \mp i\eta$, where the last equality made use of η being infinitesimal. We now perform a so-called Wick-rotation by integrating along the contour C displayed in figure 3,

$$0 = \oint_C dk_0 \frac{1}{[k_0^2 - \vec{k}^2 - m^2 + i\eta]^n}$$

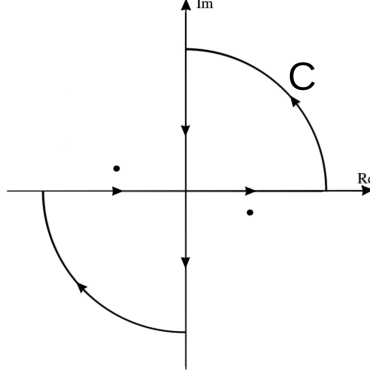


Figure 3: Graphical representation of the Wick rotation in the complex k_0 -plane. The dots denote the poles of the propagator at $k_0^\pm = \pm\sqrt{\vec{k}^2 + m^2 - i\eta}$.

and use that the contribution of the circles vanishes as the radius tends to infinity. This allows us to express $I_n(m^2)$ as

$$I_n(m^2) = i(-1)^{-n} \int \frac{d^D k_E}{(2\pi)^D} \frac{1}{[k_E^2 + (m^2 - i\eta)]^n},$$

where now $k_E^2 = (k_E^0)^2 + \vec{k}^2$ with $k_E^0 = -ik_0$ has underlying euclidean metric. Looking at the integral, we see that it depends only on k_E^2 , which tells us that it is a good idea to introduce D -dimensional spherical coordinates according to

$$d^D k_E = d\Omega_D dr r^{D-1}$$

with $r^2 = k_E^2$ and the D -dimensional solid angle $d\Omega_D$ which satisfies

$$\int d\Omega_D = \frac{2\pi^{D/2}}{\Gamma(D/2)}.$$

The final answer for $I_n(m^2)$ is

$$I_n(m^2) = \frac{i(-1)^{-n}}{(4\pi)^{D/2}} \frac{\Gamma(n - D/2)}{\Gamma(n)} (m^2 - i\eta)^{D/2-n}.$$

4.1 Classification of one-loop integrals

The notation for one-loop integrals is standardized, and the various integrals are defined as

$$\begin{aligned}
A_0(m^2) &= \frac{(2\pi\mu)^{4-D}}{i\pi^2} \int d^D k \frac{1}{[k^2 - m^2 + i\eta]} = \frac{(2\pi\mu)^{4-D}}{i\pi^2} (2\pi)^D I_1(m^2) \\
&= -m^2 \left(\frac{4\pi\mu^2}{m^2} \right)^{2-D/2} \Gamma\left(\frac{2-D}{2}\right) = -m^2 \left(\frac{4\pi\mu^2}{m^2} \right)^\epsilon \Gamma(\epsilon - 1) \\
&= m^2 \left[\underbrace{\frac{1}{\epsilon} - \gamma_E + \ln(4\pi)}_{=:\Delta_{\text{div}}} + \ln\left(\frac{\mu^2}{m^2}\right) + 1 + \mathcal{O}(\epsilon) \right]
\end{aligned}$$

where we extracted the divergent and the finite pieces explicitly.

The next integral is

$$B_0(p^2, m_1^2, m_2^2) = \frac{(2\pi\mu)^{4-D}}{i\pi^2} \int d^D k \frac{1}{[k^2 - m_1^2 + i\eta] [(k+p)^2 - m_2^2 + i\eta]}.$$

We combine the two denominators by means of Feynman parameters according to

$$\frac{1}{A^n B^m} = \frac{\Gamma(n+m)}{\Gamma(n)\Gamma(m)} \int_0^1 dx \frac{x^{n-1} (1-x)^{m-1}}{[A x + B(1-x)]^{n+m}}$$

and complete the square with respect to k . After the shift $k \rightarrow k - p(1-x)$ we can use the result for $I_n(m^2)$ to arrive at

$$\begin{aligned}
B_0(p^2, m_1^2, m_2^2) &= (4\pi\mu^2)^\epsilon \Gamma(\epsilon) \int_0^1 dx \frac{1}{[m_1^2 x + m_2^2(1-x) - p^2 x(1-x) - i\eta]^\epsilon} \\
&= \Delta_{\text{div}} - \int_0^1 dx \ln \left(\frac{m_1^2 x + m_2^2(1-x) - p^2 x(1-x) - i\eta}{\mu^2} \right) + \mathcal{O}(\epsilon).
\end{aligned}$$

For integrals with more than two denominators we can use this procedure iteratively. Note that the integral

$$\begin{aligned}
C_0(p_1^2, (p_1 - p_2)^2, p_2^2, m_1^2, m_2^2, m_3^2) &= \\
&= \frac{(2\pi\mu)^{4-D}}{i\pi^2} \int d^D k \frac{1}{[k^2 - m_1^2 + i\eta] [(k+p_1)^2 - m_2^2 + i\eta] [(k+p_2)^2 - m_3^2 + i\eta]}
\end{aligned}$$

develops an infrared (IR) divergence if one of the masses vanishes. This results in a pole in ϵ which is not captured by the `FeynCalc` package since it extracts only ultraviolet (UV) divergences (i.e. those arising from large $|k^\mu|$).

For vector and tensor integrals we adopt the so-called *Passarino-Veltman reduction*, which we outline with a few examples. We start with the vector integral

$$B^\mu = \frac{(2\pi\mu)^{4-D}}{i\pi^2} \int d^D k \frac{k^\mu}{[k^2 - m_1^2 + i\eta] [(k+p)^2 - m_2^2 + i\eta]},$$

whose Lorentz-decomposition must be of the form

$$B^\mu = p^\mu B_1$$

since after the loop integration p^μ is the only remaining four-vector the integral can depend on. We now contract the last equation with p_μ and write

$$2k \cdot p = [(k+p)^2 - m_2^2] - [k^2 - m_1^2] - (p^2 - m_2^2 + m_1^2)$$

to cancel as many denominators as possible. We then arrive at

$$2p^2 B_1 = A_0(m_1^2) - A_0(m_2^2) - (p^2 - m_2^2 + m_1^2) B_0(p^2, m_1^2, m_2^2).$$

For the tensor integral

$$B^{\mu\nu} = \frac{(2\pi\mu)^{4-D}}{i\pi^2} \int d^D k \frac{k^\mu k^\nu}{[k^2 - m_1^2 + i\eta] [(k+p)^2 - m_2^2 + i\eta]}$$

the ansatz reads

$$B^{\mu\nu} = g^{\mu\nu} B_{00} + p^\mu p^\nu B_{11}.$$

Contracting this equation first with $g_{\mu\nu}$ and subsequently with $p_\mu p_\nu$ yields the following linear system of equations for the two unknowns B_{00} and B_{11} ,

$$D B_{00} + p^2 B_{11} = A_0(m_2^2) + m_1^2 B_0(p^2, m_1^2, m_2^2)$$

$$B_{00} + p^2 B_{11} = \frac{1}{2} A_0(m_2^2) - \frac{1}{2} (p^2 - m_2^2 + m_1^2) B_1.$$

The ansatz for

$$C^\mu = \frac{(2\pi\mu)^{4-D}}{i\pi^2} \int d^D k \frac{k^\mu}{[k^2 - m_1^2 + i\eta] [(k+p_1)^2 - m_2^2 + i\eta] [(k+p_2)^2 - m_3^2 + i\eta]}$$

would read $C^\mu = p_1^\mu C_1 + p_2^\mu C_2$ and so on. In general the ansatz has to include all tensor structures that can be formed out of the independent external momenta and the metric. The package `FeynCalc` is also able to perform the Passarino-Veltman reduction. The program `package-X` [19] contains results for the one-loop integrals A_0 , B_0 , C_0 , ... for various kinematic configurations.

5 The Mathematica package FeynCalc

FeynCalc is a Mathematica package for the automated calculation of amplitudes and cross sections in quantum field theory, currently at version 9.3 [18]. The package FeynHelpers [20] connects FeynCalc and package-X, thus allowing to easily use the known one loop integrals within FeynCalc (without worrying about different conventions). We summarize some of the most frequently-used commands here, and refer to [18] and [20] for more information

- `Pair[LorentzIndex[μ], Momentum[v]]`
Denotes the four-momentum v^μ . The command `Pair` also allows to input the metric (`Pair[LorentzIndex[μ], LorentzIndex[ν]]`) and scalar products (`Pair[Momentum[u], Momentum[v]]`).
- `MetricTensor[μ , ν]`
Alternative way to input the metric.
- `Eps[LorentzIndex[μ], LorentzIndex[ν], LorentzIndex[ρ], LorentzIndex[σ]]`
Levi-Civita tensor, always assumes upper indices and uses $\epsilon^{0123} = +1$. If contracted with momenta, can write for example `Eps[LorentzIndex[mu], LorentzIndex[nu], Momentum[p], Momentum[q]]` for $\epsilon^{\mu\nu\alpha\beta} p_\alpha q_\beta$.
- `Contract[expr]`
Contracts equal indices in *expr*.
- `DiracGamma[LorentzIndex[mu]], DiracGamma[5], DiracGamma[Momentum[p]]`
Dirac matrices γ^μ , γ_5 , and \not{p} .
- `GA[mu], GA[5], GS[p]`
Like before, but in a shorter (external) notation. FCI and FCE (for FeynCalcInternal/External) switch between the two notations. Note that the two notations also have different `FullForm`.
- `GA[6], GA[7]`, short-hand notation for the projectors $P_{R/L} = (1 \pm \gamma_5)/2$.
- `GA[mu].GA[nu]` or simply `GA[mu, nu]`. Product $\gamma^\mu \gamma^\nu$ of Dirac matrices
- `DiracSigma[GA[mu, nu]]`, $\sigma^{\mu\nu}$.

- `DiracReduce[expr]`
Reduces *expr* to the basis $\{1, \gamma_5, \gamma^\mu, \gamma^\mu \gamma_5, \sigma^{\mu\nu}\}$ of bilinear covariants.
- When working in dimensional regularization, *D*-dimensional quantities have to be specified explicitly, e.g.
`Pair[LorentzIndex[μ, D], Momentum[p, D]],`
`DiracGamma[LorentzIndex[μ, D], D],`
`GAD[μ], GSD[p]` etc.
Note that if both four- and *D*-dimensional objects appear, everything is calculated in four dimensions!
- `Tr[expr]`
computes the Dirac trace of *expr*. Warning: Traces without explicit Dirac structures are not evaluated.
- There are various commands to simplify expressions, such as
`DiracSimplify, DiracReduce, Calc, DotSimplify, DiracOrder.`
- To work with loop functions we can use
`FeynAmpDenominator[PropagatorDenominator[Momentum[k, D], m]]` to input a loop propagator. The short-hand (external) notation would be `FAD[k, m]`. For several propagators, write e.g.
`FeynAmpDenominator[PropagatorDenominator[Momentum[k, D], m1], PropagatorDenominator[Momentum[k, D] + Momentum[p, D], m2]]` or `FAD[k, m1, k + p, m2]`.
- The Passarino-Veltman reduction is done with
`TID[integrand, loop momentum]`, e.g. as in
`TID[GSD[k].GAD[μ].GSD[k] FAD[k, m], k]`, i.e. it also works with non-trivial numerators. With the option `ToPaVe -> True` the command `TID` expresses the result in terms of the standard notation for one-loop integrals (A_0, B_0, \dots).
- In order to substitute the known results of the standard one-loop integrals (A_0, B_0, \dots) use
`PaXEvaluate[expression, PaXImplicitPrefactor -> fac]`. By choosing $fac = 1/(2\pi)^D$ in the option `PaXImplicitPrefactor -> fac` the correct integration measure following from the Feynman rules is used. One can also choose a different factor, for instance to get rid of $\log(4\pi)$ or Euler's constant γ_E in the results. Note that `PaXEvaluate` is a command of `FeynHelpers`, which requires a working (and correctly linked) installation of `package-X`.

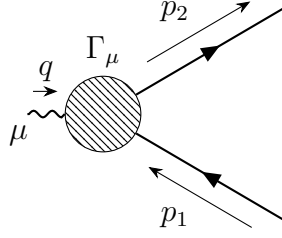


Figure 4: Diagrammatic representation of the scattering of an electron with an electromagnetic field.

The package has also more features such as calculations in the color algebra of $SU(N)$ etc. We omit these features since they are not relevant for this lab course.

6 The electron anomalous magnetic moment

The anomalous magnetic moment of the electron is extracted from the scattering of an electron with an electromagnetic field at zero momentum transfer (see figure 4). The kinematic situation therefore is

$$\begin{aligned} q^\mu &= p_2^\mu - p_1^\mu, & p_1 \cdot p_2 &= m^2 \quad (\text{since } q^2 = (p_2 - p_1)^2 = 0), \\ q^2 &= 0, & \not{p}_1 u(p_1, s_1) &= m u(p_1, s_1), \\ p_1^2 &= p_2^2 = m^2, & \bar{u}(p_2, s_2) \not{p}_2 &= \bar{u}(p_2, s_2) m. \end{aligned}$$

Using the Gordon identity

$$\bar{u}(p_2, s_2) \gamma_\mu u(p_1, s_1) = \frac{1}{2m} \bar{u}(p_2, s_2) [(p_1 + p_2)_\mu + i\sigma_{\mu\nu} (p_2 - p_1)^\nu] u(p_1, s_1)$$

for an on-shell Dirac-fermion of mass m , we can decompose the expression Γ_μ according to the above Lorentz structures and read off the g -factor of the electron,

$$\bar{u}(p_2, s_2) \Gamma_\mu u(p_1, s_1) = i \bar{u}(p_2, s_2) \left[e \frac{(p_1 + p_2)_\mu}{2m} + g_e \frac{e}{2m} \frac{\sigma_{\mu\nu}}{2} i q^\nu \right] u(p_1, s_1).$$

There is also the possibility to use projection operators together with Dirac traces to extract g_e respectively $a_e = (g_e - 2)/2$. The procedure is

outlined in [17] and includes the following steps. One first takes $q^2 \neq 0$ and decomposes the amplitude according to

$$\bar{u}(p_2, s_2) \Gamma_\mu u(p_1, s_1) = \bar{u}(p_2, s_2) \left[F_1(t) \gamma_\mu - \frac{i}{2m} F_2(t) \sigma_{\mu\nu} \Delta^\nu + \frac{1}{m} F_3(t) \Delta_\mu \right] u(p_1, s_1).$$

Note that reference [17] uses the additional notation $\Delta^\mu = -q^\mu = p_1^\mu - p_2^\mu$, $p^\mu = \frac{1}{2}(p_1^\mu + p_2^\mu)$, and $t = \Delta^2 = q^2$. One also has $p^2 = \frac{1}{4}(4m^2 - t)$ and $p \cdot \Delta = 0$.

Conservation of the electromagnetic current requires $F_3(t) = 0$. $F_1(t)$ is the charge form factor, $F_2(t)$ the magnetic moment form factor, and the anomalous magnetic moment of the electron is given by

$$a_e = (g_e - 2)/2 = -iF_2(0)/e.$$

The magnetic moment form factor $F_2(t)$ can then be extracted via [17]

$$F_2(t) = \frac{-2m^2}{(D-2)t(t-4m^2)} \times \text{Tr} \left[(\not{p}_1 + m) \left(\gamma_\mu + \frac{4m^2 + (D-2)t}{m(t-4m^2)} p_\mu \right) (\not{p}_2 + m) \Gamma^\mu \right].$$

This formula contains a factor $1/t$, and therefore we can take $t = q^2 \rightarrow 0$ only at the very end of the calculation. However, it is also possible to extract directly the anomaly $F_2(0)$. To this end the general amplitude Γ_μ is expanded to first order in Δ_μ :

$$\begin{aligned} \Gamma_\mu(p, \Delta) &\approx \Gamma_\mu(p, 0) + \Delta_\nu \frac{\partial}{\partial \Delta_\nu} \Gamma_\mu(p, \Delta) \Big|_{\Delta=0} \\ &\equiv V_\mu(p) + \Delta_\nu T_\mu^\nu(p). \end{aligned}$$

The formula for a_e then becomes

$$\begin{aligned} F_2(0) &= \frac{1}{2(D-1)(D-2)m^2} \text{Tr} \left\{ \frac{D-2}{2} [m^2 \gamma_\mu - D p_\mu \not{p} - (D-1) m p_\mu] V^\mu \right. \\ &\quad \left. + \frac{m}{4} (\not{p} + m) [\gamma_\nu, \gamma_\mu] (\not{p} + m) T^{\mu\nu} \right\}. \end{aligned}$$

7 Tasks for the lab course

The task for the lab course consist of two parts

1. **Renormalization of QED at the one-loop level in the pole scheme.** By considering the electron self-energy and the vacuum polarization, determine the renormalization constants $\delta Z_2^{(1)}$, $\delta Z_m^{(1)}$, and $\delta Z_3^{(1)}$ in the pole scheme ($\delta Z_1^{(1)}$ then follows from the Ward identity). Use the Mathematica package `FeynCalc` for your calculation. For the calculation of the self-energy follow the strategy of [21]. It can also be helpful to have a look in the lecture notes [22].
2. **Computation of the one-loop contribution to the anomalous magnetic moment of the electron.** Using Mathematica, write a program that evaluates the one-loop correction to the anomalous magnetic moment of the electron in the pole scheme. You can (but don't have to) use the packages `FeynCalc` and `package-X` for your calculation. Moreover, you can choose one of the three strategies described in section 6 to perform the calculation.

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